The Ground State Degeneracy of Two Quantum Spin Chains

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Abstract

We consider quantum spin systems with a Hilbert space V, defined as the tensor product of N finite-dimensional complex Hilbert spaces. The time evolution and energy spectrum of the system are defined by the Hamiltonian, given as a Hermitian operator acting on V. These Hamiltonians model interactions between spins within the system. The goal of the project is to study the ground states, or the lowest energy states, of the system. These are given as the unit vectors in the eigenspace belonging to the smallest eigenvalue of the Hamiltonian. In particular, we are interested in the dimension of this eigenspace, which is commonly referred to as the ground state degeneracy. We study two specific models: the Heisenberg model and the Majumdar-Ghosh model. The Heisenberg model is a sum of terms acting on a pair of nearest neighbor tensor factors. The Majumdar-Ghosh model deals with next-nearest neighbor spins, and is a sum of operators, h, which acts on three consecutive spins. The ground state degeneracy depends on N for both models. The nature of the correlation of spins depends on the different interactions within each model. The Majumdar-Ghosh model exhibits an effect called dimerization.

I. Introduction

The theory of quantum spin systems deals with properties of quantum systems with a finite number of degrees of freedom, N, each having a finite-dimensional state space. Finite systems are studied in the form of models where each degree of freedom is thought of as a spin variable. This model is defined by describing a Hamiltonian, which operates on the Hilbert space. Using this, we can study the total energy of the system, which is given by the eigenvalues of the Hamiltonian. For example, in the study of magnetism, a model in which the basic interaction given by the exchange of the states of two spins plays an important role[5].

We begin by studying the lowest energy level of these systems, otherwise known as the ground state. These are given as the unit vectors in the eigenspace belonging to the smallest eigenvalue of the Hamiltonian. In particular, we are interested in the dimension of this eigenspace, which is commonly referred to as the ground state degeneracy. We begin with the Heisenberg model,

$$H_N = \sum_{i=1}^{N} t_{i,i+1}$$

which is composed by the sum of nearest neighbor interactions on a chain of spins, (i.e. $t_{i,i+1}$ operates on two adjacent spins and exchanges their states). Bethe proposed a process to find the exact eigenvectors of spin-$\frac{1}{2}$ systems [1]. He served as a trailblazer for the study of quantum spin models, and many studies on similar systems and models are based on his work.

Majumdar and Ghosh expanded on the study of quantum spin systems by extending the Heisenberg model to also consider the interaction between next-nearest spins. They looked at the stability of the Heisenberg model when a next-nearest neighbor interaction was added, as modeled by the Hamiltonian

$$H^{MG} = \frac{1}{2}J\sum_{i=1}^{N} t_{i,i+1} + \frac{1}{2}J\alpha\sum_{i=1}^{N} t_{i,i+2}$$

where $J < 0$ indicates a ferromagnetic system and $J > 0$ indicates an antiferromagnetic one. They sought to observe the nature of any such instability in the system and what new ground states occur. They presented full results for some finite chains, including their eigenvectors and eigenvalues, and found that the systems were largely non-magnetic except when ferromagnetic in nature. For antiferromagnetic states, the ground states remain stable and always have spin-0, but with a strong antiferromagnetic next nearest neighbor interaction, an effect called symmetry breaking occurs [4].

Caspers, Emmett, and Magnus continued to study what is now known as the Majumdar-Ghosh model and proved that its previously calculated ground states, where the spins are arranged in nearest-neighbor singlet pairs, are the only ground states. A system with an even number of spins and periodic boundary conditions has two such states, while for an open boundary chain the number depends on the number of spins, N. They then looked at excited states, or higher energy levels, finding an upper bound for the lowest energy and studying "defects" in the system that compose excitations. Most importantly, they provided solid proof that there is a gap in the spectrum between states [2].

Majumdar also worked on low lying excited states, approximating them through the linearized equation of motion technique. By breaking one or more singlet pairs, one can produce the spin-1 states which indicate a higher energy state. Lieb and Mattis proved that the energy level of a system with spin S has a smaller eigenvalue that a system spin $S + 1$, thus spin-1 states are the next highest excited state for the Heisenberg model. [3]

Some systems have ground states that that exhibit well-defined physical behaviors. Dimerization is the forming of singlet pairs, or dimers, as a result of interactions between spins, most notably in the ground state. It is the most common type of spontaneous lattice translation symmetry breaking in the ground state of quantum spin systems. Nachtergaele and Ueltschi showed that it occurs for some quantum spin chains with nearest neighbor interactions and spins with sufficiently large magnitudes. It does not occur in the Heisenberg model presented in this paper, but it does in models such as the Majumdar-Ghosh model. [6].

My research focuses on deriving the ground states of these models. The Heisenberg ground states are first calculated explicitly using the canonical orthonormal basis $e_0 = \begin{pmatrix}1\\0\end{pmatrix}, e_1 = \begin{pmatrix}0\\1\end{pmatrix}$ to find the matrix representation of the Hamiltonian and finding its lowest eigenvalue. For the ferromagnetic Heisenberg model, the ground state energy is $-N$ with a degeneracy of $N+1$. I then studied the Majumdar Ghosh model, using my work from the Heisenberg model as a starting point. I found that the ground state is given by eigenvalue 0, and the degeneracy is 4 if N is odd and 5 is N is even. Here, the singlet dimers are represented by the vector $\xi = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$.

II. The Heisenberg model

First, we look at a closed boundary chain of N spins. The Hilbert space of states of this system is $V = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$. Here, we take the canonical orthonormal basis

$$e_0 = \begin{pmatrix}1\\0\end{pmatrix}, e_1 = \begin{pmatrix}0\\1\end{pmatrix}$$

as the basis of $\mathbb{C}^2$.

Let $t : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ be a linear map where $\mathbf{u} \otimes \mathbf{v} \mapsto \mathbf{v} \otimes \mathbf{u}$, so the function $t_{i,i+1} \in \mathcal{L}(V)$, which acts on a spin i and switches it with its neighbor, $i+1$ is:

$$t_{i,i+1}(e_{\sigma_1} \otimes \cdots \otimes e_{\sigma_N}) = e_{\sigma_1} \otimes \cdots \otimes e_{\sigma_{i+1}} \otimes e_{\sigma_i} \otimes \cdots \otimes e_{\sigma_N}$$

It is represented by a hermitian, real symmetric matrix:

$$[t] = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\end{pmatrix}$$

that has eigenvalues 1 and -1. When t is acting on a ring of N spins, there are $3 * 2^{N-2}$ eigenvectors corresponding to eigenvalue 1, and $1 * 2^{N-2}$ corresponding to -1. (e.g. in a system of 2 spins, there are three eigenvectors of eigenvalue 1 and one eigenvector of eigenvalue -1).

The the first Hamiltonian operator we studied is the antiferromagnetic Heisenberg model,

$$H_N = \sum_{i=1}^{N} t_{i,i+1}$$

a hermitian, $2^N \times 2^N$ matrix in $\mathcal{L}(V)$. It acts on the whole chain, and changes the states of two adjacent spins while keeping the others as is.

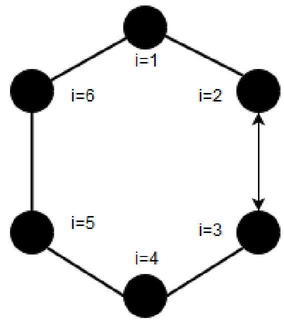

Figure 1: A system of $N = 6$ spins acted on by the operator t_{23}

Claim 1. *For hermitian matrices, the largest possible eigenvalue of their sum is at most the sum of the largest eigenvalues of each matrix (i.e if λ and μ are the largest eigenvalue of matrices A and B respectively, the largest possible eigenvalue of $A + B$ is $\lambda + \mu$).*

Proof. For all $\mathbf{v} \in V$, we know that $\langle \mathbf{v}, A\mathbf{v} \rangle \leq \lambda \langle \mathbf{v}, \mathbf{v} \rangle$ and there exists an eigenvector of A, $\mathbf{v}_0 \neq 0$, such that $\langle \mathbf{v}_0, A\mathbf{v}_0 \rangle = \lambda \langle \mathbf{v}_0, \mathbf{v}_0 \rangle$.

Then we have

$$\begin{aligned} \langle \mathbf{v}, (A+B)\mathbf{v} \rangle &= \langle \mathbf{v}, A\mathbf{v} + B\mathbf{v} \rangle \\ &\leq \langle \mathbf{v}, A\mathbf{v} \rangle + \langle \mathbf{v}, B\mathbf{v} \rangle \\ &\leq \lambda \langle \mathbf{v}, \mathbf{v} \rangle + \mu \langle \mathbf{v}, \mathbf{v} \rangle \\ &\leq (\lambda + \mu) \langle \mathbf{v}, \mathbf{v} \rangle \end{aligned}$$

which implies that the largest eigenvalue of $A + B$ is less than or equal to $\lambda + \mu$.

□

Thus the largest possible eigenvalue for H_N is N, since the largest possible eigenvalue of $t_{i,i+1}$ is 1. For any eigenvector, v_0, of H_N corresponding to eigenvalue N, we must have $t_{i,i+1}(v_0) = v_0$ for all i.

Let $M : V \to V$ be a linear transformation in $\mathcal{L}(V)$ given by the equation:

$$M = \sum_{i=1}^{n} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_i$$

which counts the number of 1's in a basis vector. Thus where $e_{\sigma_1 \sigma_2 \ldots \sigma_N} = \{e_{\sigma_1} \otimes \cdots \otimes e_{\sigma_N} | \sigma_j \in \{0, 1\}\}$,

$$M(e_{\sigma_1 \sigma_2 \ldots \sigma_N}) = (\sum_{i=1}^{N} \sigma_j)(e_{\sigma_1 \sigma_2 \ldots \sigma_N})$$

The matrix M is diagonal in the tensor product basis. Its eigenvalues are the numbers of ones that label $e_{\sigma_1 \sigma_2 \ldots \sigma_N}$ in the basis of the eigenspace of M.

The two models, H_N and M commute.

Proof. $[H_N, M] = H_N M - M H_N = \sum_{i=1}^{N} t_{i,i+1} M - \sum_{i=1}^{N} M t_{i,i+1} = \sum_{i=1}^{N} [t_{i,i+1}, M] = 0$ since H_N just commutes the zeroes and ones. So when M is applied first, the value of the product does not change. □

Let V_λ be a linear subspace of V containing all eigenvectors of M corresponding to eigenvalue λ, so $V_\lambda = \{\mathbf{v} \in V | M\mathbf{v} = \lambda \mathbf{v}\}$.

Proposition 1. *Let λ be an eigenvalue of M with eigenspce V_λ. V_λ is an invariant subspace for H_N.*

Proof. Let $\mathbf{w} = H_N \mathbf{v}$ for any $\mathbf{v} \in V_\lambda$. We know that $M\mathbf{v} = \lambda\mathbf{v}$. Using the fact that H_N and M commute,

$$\begin{aligned} M\mathbf{w} &= M(H_N \mathbf{v}) \\ &= (M H_N)\mathbf{v} \\ &= (H_N M)\mathbf{v} \\ &= H_N (M\mathbf{v}) \\ &= H_N (\lambda \mathbf{v}) \\ &= \lambda H_N \mathbf{v} \\ &= \lambda \mathbf{w} \end{aligned}$$

Hence, $\mathbf{w} \in V_\lambda$. □

Moreover, the eigenvalues of M are $\lambda = 0, 1, \ldots, N$.
The direct sum of all these subspaces V_λ make up the whole space V, so:

$$V = \bigoplus_{\lambda=0}^{N} V_\lambda$$

Since the V_λ are invariant subspaces for H_N, H_N can be written as a $2^N \times 2^N$ block diagonal matrix. By the above theorem, this can be done because it commutes with M. We denote $G_\lambda = H_N|_{V_\lambda}$ as each block of this matrix, corresponding to eigenvalue λ of M:

$$H_N = \begin{pmatrix} G_0 & & & \\ & G_1 & & \\ & & \ddots & \\ & & & G_N \end{pmatrix}$$

The blocks have dimension $D_\lambda \times D_\lambda$, where

$$D_\lambda = \binom{N}{\lambda} = \frac{N!}{\lambda!(N-\lambda)!}$$

Blocks G_0 and G_N are 1×1 matrices with value N, thus they have eigenvalue N.

Call $E_i^{(N)}$, $i = 0, 1, \ldots N$, the eigenvalues of H_N with $E_0^{(N)} < E_1^{(N)} < \ldots < E_N^{(N)} = N$. The multiplicities for any $E_i^{(N)}$ can be denoted as eigenvectors $d_{ij}^{(N)}$, $j = 0, 1, \ldots$.

For any eigenvector $\mathbf{v} = \sum e_{\sigma_1\sigma_2\ldots\sigma_N}$, $\sigma_i \in \{0,1\}$, let k be the number of zeroes and $N-k$ be the number of ones, so $N-k = \sum_{i=1}^{N} \sigma_i$. Recall for eigenvalue N of H_N, we have that $h_{i,i+1}(\mathbf{v}) = \mathbf{v}$, $\forall i = 1,2,\ldots N$. For any such $\mathbf{v}$, k has possible values $0,1,\ldots N$, thus there are $N+1$ eigenvectors for eigenvalue N in H_N.

The ferromagnetic Heisenberg model is denoted by $H_N^F = -H_N$, and is what is called a frustration free model. This means that its ground state is the sum of the ground states of its component operators. The antiferromagnetic Heisenberg model is not frustration free, making it more difficult to calculate its ground state. However, as we see that the eigenvalues λ of H_N^F are just the negative of the eigenvalues of H_N, the ground state of H_N^F is represented by the eigenvalue $\lambda = -N$, and also has a degeneracy of $N+1$.

III. Majumdar-Ghosh model

Now we look at the open boundary condition, where we have an open-ended chain of N spins, as opposed to a closed ring. Here we denote the orthonormal basis of $\mathbb{C}^2$ as $\{|+\rangle, |-\rangle\}$. The Majumdar-Ghosh model looks at the interaction between next nearest spins. For a chain of N spins, we denote this Hamiltonian as

$$H_N^{MG} = h_{123} \otimes \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{N-3} + \mathbb{1} \otimes h_{234} \otimes \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{N-4} + \cdots + \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{N-3} \otimes h_{N-2N-1N}$$

where $h_{ijk} = t_{ij} + t_{jk} + t_{ik}$. This Hamiltonian also lies in the vector space $\mathcal{L}(V)$, so $dim(H^{MG}) = 2^N \times 2^N$. Its ground state is the eigenvalue 0 (another way to think of this is the kernel of the Hamiltonian).

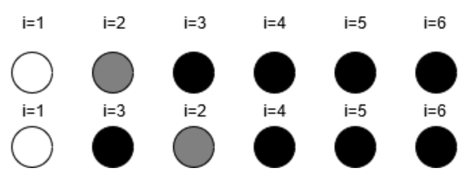

Figure 2: An open boundary system of $N = 6$ spins acted on by the operator h_{123}

For symmetric states, or when the Hamiltonian does not affect the positions of the spins, each t_{xy} has eigenvalue 1, and for anti-symmetric states, it has eigenvalue -1. The vector $\xi = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2$ is an example of an antisymmetric vector, as $H_2(\xi) = -\xi$. Note that for any $\mathbf{u} \in \mathbb{C}^2$, $H_3^{MG}(\xi \otimes \mathbf{u}) = 0$ and $H_3^{MG}(\mathbf{u} \otimes \xi) = 0$, and that $H_4^{MG}(\xi \otimes \xi) = 0$. From there we claim that the ground state degeneracy of open boundary systems with an even N is 5, and is 4 for systems with an odd N.

To prove this we need two important facts:

Lemma 1. *Let let h be an arbitray Hamiltonian acting on a system of N spins occupying a Hilbert space $V = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{N}$. Say $K = \ker(h) \subset \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{N}$. Then $\ker(h \otimes \mathbb{1}) = K \otimes \mathbb{C}^2$.*

Proof. First we want to show that $\ker(h \otimes \mathbb{1}) \subset K \otimes \mathbb{C}^2$.

We have that $K \otimes \mathbb{C}^2 = \{w_+ \otimes |+\rangle + w_- \otimes |-\rangle \mid w_+, w_- \in K\}$. For all $\mathbf{v} \in \ker(h \otimes \mathbb{1}) \subseteq \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{N} \Rightarrow \exists v_+, v_- \in \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{N-1}$ such that $\mathbf{v} = v_+ \otimes |+\rangle + v_- \otimes |-\rangle$. We know that $(h \otimes \mathbb{1})\mathbf{v} = 0$. Thus,

$$\begin{aligned}
&(h \otimes \mathbb{1})\mathbf{v} = 0 \\
&\Rightarrow (h \otimes \mathbb{1})(v_+ \otimes |+\rangle + v_- \otimes |-\rangle) = 0 \\
&\Rightarrow (h \otimes \mathbb{1})(v_+ \otimes |+\rangle) + (h \otimes \mathbb{1})(v_- \otimes |-\rangle) = 0 \\
&\Rightarrow h(v_+) \otimes |+\rangle + h(v_-\rangle \otimes |-\rangle = 0
\end{aligned}$$

For all $a, b \neq 0 \in \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{N-1}$, $a \otimes |+\rangle$ and $b \otimes |-\rangle$ being linearly independent tells us that $a \otimes |+\rangle + b \otimes |-\rangle = 0 \Rightarrow a = b = 0$. Therefore,

$$\begin{aligned}
&h(v_+) \otimes |+\rangle + h(v_-\rangle \otimes |-\rangle = 0 \\
&\Rightarrow h(v_+) \otimes |+\rangle = 0, h(v_-) \otimes |-\rangle = 0 \\
&\Rightarrow h(v_+) = 0, h(v_-) = 0 \\
&\Rightarrow v_+, v_- \in K.
\end{aligned}$$

To show that $K \otimes \mathbb{C}^2 \subset \ker(h \otimes \mathbb{1})$, take $\mathbf{w} = w_+ \otimes |+\rangle + w_- \otimes |-\rangle \in K \otimes \mathbb{C}^2$.

$$\begin{aligned}
(h \otimes \mathbb{1})(w_+ \otimes |+\rangle + w_- \otimes |-\rangle) &= (h \otimes \mathbb{1})(w_+ \otimes |+\rangle) + (h \otimes \mathbb{1})(w_- \otimes |-\rangle) \\
&= h(w_+) \otimes |+\rangle + h(w_-) \otimes |-\rangle \\
&= 0
\end{aligned}$$

Thus, $\mathbf{w} \in \ker(h \otimes \mathbb{1})$.

□

Lemma 2. *Assume E and F to be non-negative definite so that $H = E + F$ is also non-negative definite (i.e. $H, E, F \geq 0$). Then* $\ker H = \ker E \cap \ker F$.

Proof. 1) First, to show that $\ker E \cap \ker F \subset \ker H$, take $\mathbf{w}$ such that $E\mathbf{w} = 0$ and $F\mathbf{w} = 0$. Then we see that $H\mathbf{w} = (E + F)\mathbf{w} = E\mathbf{w} + F\mathbf{w} = 0 + 0 = 0$.

2) Now we want to show $\ker H \subset \ker E \cap \ker F$.
For $\mathbf{v} \in kerH$,

$$\begin{aligned}
&H\mathbf{v} = 0 \\
&\Rightarrow \|H\mathbf{v}\|^2 = 0 \\
&\Leftrightarrow \langle H\mathbf{v}, H\mathbf{v}\rangle = 0 \\
&\Leftrightarrow \langle (E + F)\mathbf{v}, (E + F)\mathbf{v}\rangle = 0 \\
&\Rightarrow \langle E\mathbf{v}, E\mathbf{v}\rangle + \langle E\mathbf{v}, F\mathbf{v}\rangle + \langle F\mathbf{v}, E\mathbf{v}\rangle + \langle F\mathbf{v}, F\mathbf{v}\rangle = 0
\end{aligned}$$

only if $F\mathbf{v} = cE\mathbf{v}$ for some $c \in \mathbb{C}$. We claim that $c \geq 0$.

This is proven through the non-negative definiteness of E and F, as $0 \leq \langle \mathbf{v}, F\mathbf{v} \rangle = \langle \mathbf{v}, cE\mathbf{v} \rangle = c\langle \mathbf{v}, E\mathbf{v} \rangle$. Since $\langle \mathbf{v}, E\mathbf{v} \rangle \geq 0$, if $\langle \mathbf{v}, E\mathbf{v} \rangle \neq 0$, then $c \geq 0$ and if $\langle \mathbf{v}, E\mathbf{v} \rangle = 0$, then c is negligible.

Now we claim that $\langle \mathbf{v}, E\mathbf{v} \rangle = 0 \Rightarrow E\mathbf{v} = 0$. Since E is non-negative definite, then for all vectors $\mathbf{w} \in W$, W being an n-dimensional complex vector space, there exsists a non-negative definite matrix B such that $B^2 = E$. We may then define $\sqrt{E} = B$. Since E is non-negative definite, it is hermetian and its eigenvalues λ_i are non-negative, $i = 1, \ldots, n$. Since it is hermetian, there exists a unitary matrix U such that $E = UDU^*$, where

$$D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

Define $B = U\sqrt{D}U^*$. $B^2 = E$, as $B^2 = U\sqrt{D}U^*U\sqrt{D}U^* = U\sqrt{D}\sqrt{D}U^* = UDU^* = E$. Now choose $\mathbf{w}$ such that $\langle \mathbf{w}, E\mathbf{w} \rangle = 0$. Then we get

$$\begin{aligned} \langle \mathbf{w}, E\mathbf{w} \rangle &= \langle \mathbf{w}, B^2\mathbf{w} \rangle \\ &= \langle \mathbf{w}, BB\mathbf{w} \rangle \\ &= \langle B\mathbf{w}, F\mathbf{w} \rangle \\ &= \|B\mathbf{w}\|^2 = 0 \\ &\Rightarrow B\mathbf{w} = 0 \\ &\Rightarrow B\mathbf{w} = 0 \\ &\Rightarrow B^2\mathbf{w} = 0 \\ &\Rightarrow E\mathbf{w} = 0 \end{aligned}$$

thus proving the claim.

Continuing with the proof of the lemma, we have that

$$\begin{aligned} &\langle E\mathbf{v}, E\mathbf{v} \rangle + \langle E\mathbf{v}, F\mathbf{v} \rangle + \langle F\mathbf{v}, E\mathbf{v} \rangle + \langle F\mathbf{v}, F\mathbf{v} \rangle = 0 \\ &\Leftrightarrow \langle E\mathbf{v}, E\mathbf{v} \rangle + c\langle E\mathbf{v}, E\mathbf{v} \rangle + \bar{c}\langle E\mathbf{v}, E\mathbf{v} \rangle + |c|^2\langle E\mathbf{v}, E\mathbf{v} \rangle = 0 \\ &\Leftrightarrow |c+1|^2\langle E\mathbf{v}, E\mathbf{v} \rangle = 0 \\ &\Rightarrow E\mathbf{v} = 0 \\ &\Rightarrow F\mathbf{v} = 0 \end{aligned}$$

since $|c+1|^2 \neq 0$.

□

We can now prove our claim about the ground state degeneracy of the open boundary Majumdar-Ghosh model.

Theorem 1. *Let G_k denote the ground state of a system of k spins. If k is even, then the ground state degeneracy of an open boundary system when acted on with the Majumdar-Ghosh model is 5. If k is odd, the ground state degeneracy is 4.*

Proof. The ground state for a system of 3 spins is modeled as $G_3 = span[\{\mathbb{C}^2 \otimes \xi\} \cup \{\xi \otimes \mathbb{C}^2\}]$, and has dimension 4.
Nexr, we look at the ground state of a system with 4 spins. We will show that the ground state of this system,

$$G_4 = (G_3 \otimes \mathbb{C}^2) \cap (\mathbb{C}^2 \otimes G_3) \tag{1}$$

is equal to a subspace of dimension 5,

$$span[\{\xi \otimes \xi\} \cup \{\mathbb{C}^2 \otimes \xi \otimes \mathbb{C}^2\}] \tag{2}$$

1) First we want to prove the inclusion (2) $\subseteq$ (1).

We note that

$$span\{\xi \otimes \xi\} \subseteq \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \Rightarrow span\{\xi \otimes \xi\} \subseteq \underbrace{[\xi \otimes \mathbb{C}^2]}_{\subseteq G_3} \otimes \mathbb{C}^2$$

Similarly we can show that $span\{\xi \otimes \xi\} \subseteq \mathbb{C}^2 \otimes G_3$. Therefore $span\{\xi \otimes \xi\} \subseteq G_4$.

Now we show that the same is true for $span\{\mathbb{C}^2 \otimes \xi \otimes \mathbb{C}^2\}$.

Let $\alpha, \beta \in \mathbb{C}^2$ represent all linear combinations of the basis vectors $|+\rangle, |-\rangle$ of $\mathbb{C}^2$. By construction, we have that

$$\alpha \otimes \xi \otimes \beta \subseteq [span(\xi \otimes \mathbb{C}^2) \otimes \mathbb{C}^2] \cup [span(\mathbb{C}^2 \otimes \xi) \otimes \mathbb{C}^2] = G_3 \otimes \mathbb{C}^2$$

Similarly, we can show that $\alpha \otimes \xi \otimes \beta \in \mathbb{C}^2 \otimes G_3$. Thus we have shown that both parts of (2) are included in G_4.

2) To show that (1) $\subseteq$ (2) we want to prove that there exist coefficients $a, b_{\pm\pm} \in \mathbb{C}$ such that all $\psi \in G_4$ have the form

$$\begin{aligned}\psi = a(\xi \otimes \xi) &+ b_{++}(|+\rangle \otimes \xi \otimes |+\rangle)\\ &+ b_{+-}(|+\rangle \otimes \xi \otimes |-\rangle)\\ &+ b_{-+}(|-\rangle \otimes \xi \otimes |+\rangle)\\ &+ b_{--}(|-\rangle \otimes \xi \otimes |-\rangle)\end{aligned}$$

and is in (2).

Let $\alpha_{\pm\pm}, \beta_{\pm\pm}, \delta_{\pm\pm}, \gamma_{\pm\pm} \in \mathbb{C}$. We have two vectors, $\psi^{(1)} \in (G_3 \otimes \mathbb{C}^2)$ and $\psi^{(2)} \in (\mathbb{C}^2 \otimes G_3)$ of the form:

$$\begin{aligned}\psi^{(1)} =& \xi \otimes [\alpha_{++}|++\rangle + \alpha_{+-}|+-\rangle + \alpha_{-+}|-+\rangle + \alpha_{--}|--\rangle]\\ &+ \beta_{++}(|+\rangle \otimes \xi \otimes |+\rangle)\\ &+ \beta_{+-}(|+\rangle \otimes \xi \otimes |-\rangle)\\ &+ \beta_{-+}(|-\rangle \otimes \xi \otimes |+\rangle)\\ &+ \beta_{--}(|-\rangle \otimes \xi \otimes |-\rangle)\\ \psi^{(2)} =& [\gamma_{++}|++\rangle + \gamma_{+-}|+-\rangle + \gamma_{-+}|-+\rangle + \gamma_{--}|--\rangle] \otimes \xi\\ &+ \delta_{++}(|+\rangle \otimes \xi \otimes |+\rangle)\\ &+ \delta_{+-}(|+\rangle \otimes \xi \otimes |-\rangle)\\ &+ \delta_{-+}(|-\rangle \otimes \xi \otimes |+\rangle)\\ &+ \delta_{--}(|-\rangle \otimes \xi \otimes |-\rangle)\end{aligned}$$

We take vectors $\phi \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, and when we compute their inner product with $\psi^{(1)}$ and $\psi^{(2)}$, the results will create equivalencies between our coefficients that will allow us to find our a and $b_{\pm\pm}$'s.

For example:

$$\begin{aligned}
\langle + + + - |\psi^{(1)}\rangle &= 0 \\
\langle + + - + |\psi^{(2)}\rangle &= \frac{1}{\sqrt{2}}\gamma_{++} \\
&\Rightarrow \gamma_{++} = 0 \\
\langle + + - + |\psi^{(1)}\rangle &= \frac{1}{\sqrt{2}}\beta_{++} \\
\langle + + - + |\psi^{(2)}\rangle &= \frac{1}{\sqrt{2}}\delta_{++} - \frac{1}{\sqrt{2}}\gamma_{++} = \frac{1}{\sqrt{2}}\delta_{++} \\
&\Rightarrow \delta_{++} = \beta_{++}
\end{aligned}$$

Through this method, we find

$$\begin{aligned}
\psi^{(1)} = \xi \otimes [\alpha_{+-}| + -\rangle - \alpha_{+-}| - +\rangle] &+ \beta_{++}(|+\rangle \otimes \xi \otimes |+\rangle) \\
&+ \beta_{+-}(|+\rangle \otimes \xi \otimes |-\rangle) \\
&+ \beta_{-+}(|-\rangle \otimes \xi \otimes |+\rangle) \\
&+ \beta_{--}(|-\rangle \otimes \xi \otimes |-\rangle) \\
\psi^{(2)} = [\alpha_{+-}| + -\rangle - \alpha_{+-}| - +\rangle] \otimes \xi &+ \beta_{++}(|+\rangle \otimes \xi \otimes |+\rangle) \\
&+ \beta_{+-}(|+\rangle \otimes \xi \otimes |-\rangle) \\
&+ \beta_{-+}(|-\rangle \otimes \xi \otimes |+\rangle) \\
&+ \beta_{--}(|-\rangle \otimes \xi \otimes |-\rangle)
\end{aligned}$$

When $\alpha_{+-} = \frac{1}{\sqrt{2}}$, $\psi^{(1)} = \psi^{(2)}$ and $\psi^{(1)}, \psi^{(2)} \in G_4$. Thus $a = \{0, 1\}$ and $b_{\pm\pm} = \beta_{\pm\pm}$ lets $\psi \in span[\{\xi \otimes \xi\} \cup \{\mathbb{C}^2 \otimes \xi \otimes \mathbb{C}^2\}]$. The theorem is thus proved for the case $k = 4$.

When we look at at a system with one more spin than the last, we show the case where a spin is added on each side of the chain, so using Lemmas 1 and 2, we see that its ground state is

$$\begin{aligned}
G_{k+1} &= \ker([H_k^{MG} \otimes \mathbb{1}] + [\mathbb{1} \otimes H_k^{MG}]) \\
&= \ker(H_k^{MG} \otimes \mathbb{1}) \cap \ker(\mathbb{1} \otimes H_k^{MG}) \\
&= (\mathbb{C}^2 \otimes G_k) \cap (G_k \otimes \mathbb{C}^2)
\end{aligned}$$

If k is odd,

$$
\begin{aligned}
G_{k+1} = &(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \bigotimes_{i=1}^{\frac{k-1}{2}} \xi) \\
&\cap (\mathbb{C}^2 \otimes \bigotimes_{i=1}^{\frac{k-1}{2}} \xi \otimes \mathbb{C}^2) \\
&\cap (\mathbb{C}^2 \otimes \bigotimes_{i=1}^{\frac{k-1}{2}} \xi \otimes \mathbb{C}^2) \\
&\cap (\bigotimes_{i=1}^{\frac{k-1}{2}} \xi \otimes \mathbb{C}^2 \otimes \mathbb{C}^2) \\
&= (\mathbb{C}^2 \otimes \bigotimes_{i=1}^{\frac{k-3}{2}} \xi \otimes \mathbb{C}^2) \cap (\bigotimes_{i=1}^{\frac{k+1}{2}} \xi)
\end{aligned}
$$

which has degeneracy equal to 5 and follows from the fact that $span\{\xi \otimes \xi\} \subseteq \xi \otimes \mathbb{C}^2$ and $\subseteq \mathbb{C}^2 \otimes \xi$.
If k is even,

$$
\begin{aligned}
G_{k+1} = &(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \bigotimes_{i=1}^{\frac{k}{2}-1} \xi \otimes \mathbb{C}^2) \\
&\cap (\mathbb{C}^2 \otimes \bigotimes_{i=1}^{\frac{k}{2}-1} \xi \otimes \bigotimes_{i=1}^{\frac{k}{2}-1} \xi) \\
&\cap (\mathbb{C}^2 \otimes \bigotimes_{i=1}^{\frac{k}{2}-1} \xi \otimes \mathbb{C}^2 \otimes \mathbb{C}^2) \\
&\cap (\bigotimes_{i=1}^{\frac{k}{2}-1} \xi \otimes \bigotimes_{i=1}^{\frac{k}{2}-1} \xi \otimes \mathbb{C}^2) \\
&= (\bigotimes_{i=1}^{\frac{k}{2}} \xi \otimes \mathbb{C}^2) \cap (\mathbb{C}^2 \otimes \bigotimes_{i=1}^{\frac{k}{2}} \xi)
\end{aligned}
$$

which has degeneracy equal to 4 and follows from the same fact as before.

□

IV. Discussion

As stated before, dimerization does not occur in the Heisenberg model, which models nearest neighbor interaction. It does occur in the next nearest neighbor interactions modeled by the Majumdar Ghosh model in the ground state. The dimers in this case are the ξ's, and the spins that do not form dimers are in the span of $\mathbb{C}^2$.

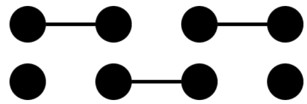

Figure 3: The ground state vectors $(\xi \otimes \xi, \mathbb{C}^2 \otimes \xi \otimes \mathbb{C}^2)$ of the Majumdar-Ghosh model with $N = 4$ spins, shown physically.

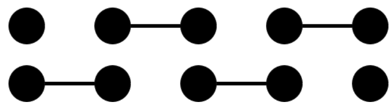

Figure 4: The ground state vectors $(\xi \otimes \xi \otimes \mathbb{C}^2, \mathbb{C}^2 \otimes \xi \otimes \xi)$ of the Majumdar-Ghosh model with $N = 5$ spins, shown physically.

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